Take home exam Geometry 7-4-2020

Always motivate your answers. You can freely use the results from the lecture notes. Please hand in your solutions in latex, scans of handwritten solutions will not be accepted. In addition the board of examiners has asked me to have you print, read, sign and scan the following declaration. Please send the signed declaration and your solutions to our email address `meetkunde20@gmail.com` before the deadline 21-4, 11pm. If some question is unclear or you believe there might be a typo, do not hesitate to contact us.
Good luck!

Declaration of the Board of Examiners

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Here are the provisions that are relevant to you sitting the exam:
1. You are required to sign the attached pledge, swearing that your work has been completed autonomously and using only the tools and aids that the examiner has allowed you to use.
2. Attempts at cheating, fraud or plagiarism will be seen as attempts to take advantage of the Corona crisis and will be dealt with very harshly by the board of examiners.
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I, ...

(enter your name and student number here)

have completed this exam myself and without help from others unless expressly allowed by my lecturer. I have come up with these answers myself. I understand that my fellow students and my lecturers are all doing their best to do their work as well as possible under the unusual circumstances of the Corona pandemic, and that any attempt by myself or my fellow students to use these circumstances to get away with cheating would be undermining those efforts and the necessary trust that this moment calls for.

Signature:

Total number of points: 90 plus two bonus points for 5i. Final grade is (10+total points)/10.

1. In this exercise we consider an affine space $\mathcal{E}$ of dimension 2020.

   (a) (5 pts) Imagine three distinct parallel affine hyperplanes $\mathcal{H}, \mathcal{H}', \mathcal{H}'' \subset \mathcal{E}$ together with two affine lines $\mathcal{D}_1, \mathcal{D}_2$ whose directions are not contained in the direction of H of $\mathcal{H}$. Define the six points $P_i = \mathcal{H} \cap \mathcal{D}_i$, $P_i' = \mathcal{H}' \cap \mathcal{D}_i$ and $P_i'' = \mathcal{H}'' \cap \mathcal{D}_i$ for $i \in \{1, 2\}$. Prove that $\frac{\overrightarrow{P_1P_1''}}{\overrightarrow{P_1P_1'}} = \frac{\overrightarrow{P_2P_2''}}{\overrightarrow{P_2P_2'}}$.

   If $\lambda\overrightarrow{P_1P_1'} = \overrightarrow{P_1P_1''}$ then there must be some $Q \in \mathcal{E}$ such that $\lambda\overrightarrow{P_2P_2'} = \overrightarrow{P_2Q}$. Since $P_2, P_2' \in \mathcal{D}_2$ this equality already shows $Q \in \mathcal{D}_2$. To finish the proof we argue that also $Q \in \mathcal{H}''$ because

$$\overrightarrow{QP_1''} = \overrightarrow{QP_2} + \overrightarrow{P_2P_1} + \overrightarrow{P_1P_1''} = -\lambda\overrightarrow{P_2P_2'} + \overrightarrow{P_2P_1} + \lambda\overrightarrow{P_1P_1'}$$

$$= \overrightarrow{P_2P_1} + \lambda(-\overrightarrow{P_2P_2'} + \overrightarrow{P_1P_2} + \overrightarrow{P_2P_2'} + \overrightarrow{P_2'P_1'}) \in H$$

   Therefore $Q = P_2''$.

   (b) (4 pts) Give an example of an affine map $\phi : \mathcal{E} \to \mathcal{E}$ that has no fixed points and is not a translation.
   For some non-zero vector $v \in E$ and $O \in \mathcal{E}$ we define $\phi(O)$ by $\overrightarrow{O\phi(O)} = v$. Next choose a linear hyperplane H such that v and H span E. For any $P \in \mathcal{E}$ there is a unique a such that $\overrightarrow{OP} = av + h$ for some $h \in H$. Define $\phi(P)$ by $\overrightarrow{\phi(O)\phi(P)} = av$. This is an affine map with underlying linear map the projection onto span v in the direction of H. It is not a bijection so also not a translation, because t_hO and O are both sent to $\phi(O)$ for any $h \in H$. It has no fixed point because a fixed point must be on the line $O\phi(O)$ but the restriction to this affine line is a non-trivial translation.

   (c) (4 pts) If $\mathcal{F}_1, \mathcal{F}_2$ are affine subspaces of $\mathcal{E}$ of equal dimension, is it true that there must exist an affine map $\phi : \mathcal{E} \to \mathcal{E}$ such that $\phi(\mathcal{F}_1) = \mathcal{F}_2$? Prove or give a counter example.

   Choose $O_i \in \mathcal{F}_i$ and define an affine map $\phi : \mathcal{E} \to \mathcal{E}$ by $\phi(O_1) = O_2$. To complete the definition of ϕ, find a linear map $f : E \to E$ that sends F_1 to F_2 (just augment bases of F_i to bases of E). Then define $\phi(P)$ by $\overrightarrow{\phi(O_1)\phi(P)} = f(\overrightarrow{O_1P})$.

2. In the Euclidean affine plane $\mathcal{E}$ consider a circle F and two distinct points A, B on F.

   (a) (4 pts) For a point C on F consider the centroid G and orthocenter H of triangle $[A, B, C]$. If I is the midpoint of $[A, B]$ and O is the center of F then use the dilation $h_{G,-1/2}$ to prove $\overrightarrow{CH} = 2\overrightarrow{OI}$.

By Theorem 3.3 we have $d_{G,-\frac{1}{2}}(H) = O$. Also CH is parallel to OI because both are perpendicular to AB. Apply Lemma 2.4 with $O, X, X', Y, Y' = G, H, C, I, O$ to see that $d_{G,-\frac{1}{2}}(I) = C$ and so $\overrightarrow{CH} = \overrightarrow{CG} + \overrightarrow{GH} = 2\overrightarrow{GI} + 2\overrightarrow{OG} = 2\overrightarrow{OI}$.

(b) (4 pts) Define a function $f : F \to \mathcal{E}$ by setting $f(C)$ to be the orthocenter of triangle $[A, B, C]$. Prove that $f(F)$ equals the reflection of F in the line AB.

Actually $f(F)$ is just the translation of F by vector $2\overrightarrow{OI}$ by the previous item. The translation can be factored into two reflections, one reflection in AB and first a reflection in the line through O perpendicular to OI. The first reflection fixes F while the second reflects F in AB as promised.

(c) (5 pts) Apply Theorem 4.1 from the lecture notes to prove that if $a = d(B, C)$ is the length of the side opposite A of triangle $[A, B, C]$ inscribed in F (whose radius is R) and α is the measure of the geometric angle opposite to a then $\frac{a}{\sin \alpha} = 2R$.

$\sin \alpha = d(P, C)/d(O, C)$ where P is the midpoint of $[B, C]$ since we know OP is the perpendicular bisector of $[B, C]$ and Theorem 4.1 from the notes says that $\angle OPC = \angle ABC$. Now $d(P, C) = a/2$ and $d(O, C) = R$ so we are done.

3. Suppose E is a three-dimensional Euclidean vector space. By a half-twist we mean is a linear isometry of E that is a rotation whose oriented angle is the flat angle.

(a) (4 pts) Prove that if s_J is the reflection in plane J through the origin then $-s_J$ is a half-twist. What is its axis?

Any vector on line $J^{\perp}$ is fixed by $-s_J$ since s_J sends it to its negative. Next, any $j \in J$ is sent to $-j$ since $s_J(j) = j$. This means $-s_J$ is a half-twist with axis $J^{\perp}$.

(b) (5 pts) Prove that for any two half-twists h, k there is an element $g \in O^+(E)$ such that $h = g \circ k \circ g^{-1}$.

Suppose a, b are the axes of h and k. In the plane spanned by a, b there exists a rotation sending b to a. Extending this by the identity in the orthogonal direction we find a rotation g sending b to a. Now we claim $h = g \circ k \circ g^{-1}$ because both sides fix all points of a and act as -1 in the orthogonal direction.

(c) (4 pts) Show that any $f \in O^+(E)$ may be written as a composition of finitely many half-twists.
By Theorem 3.1 reflections generate $O^+(E)$ and the determinant makes sure we always need an even number of reflections. By part a) $-s_J$ is a half-twist so inserting an even number of minus signs into the reflections gives the desired conclusion.

4. We work in affine three-dimensional Euclidean space $\mathcal{E}$.

(a) (3 pts) Show that the line segments connecting the mid-points of the six edges of a regular tetrahedron are precisely the edges of a regular octahedron.

A regular octahedron is formed by the midpoints of the faces of a cube. Drawing six diagonals in faces of that cube in such a way that only non-adjacent points on the cube are connected yields a tetrahedron. It must be a regular tetrahedron because the symmetries of the cube that preserve the two maximal sets of non-adjacent points transfer to the tetrahedron constructed.

(b) (3 pts) Explain how $\mathcal{E}$ is the union of countably many regular octahedra and tetrahedra of side length 1 in such a way that the intersection between any two polyhedra is empty, a single point, a single edge or a single face.

Cubes with side length 1 fill $\mathcal{E}$ in the desired way. Now decompose each cube into a regular tetrahedron as above an connect the irregular pieces in groups of eigth to make an octahedron.

(c) (4 pts) If T is a regular tetrahedron with side length 1 and centroid O and $T' = h_{O,-1}(T)$ then describe the convex hull of $T \cap T'$ and also the convex hull of $T \cup T'$.

We know that the tetrahedra T and T' are convex and so their intersection is also convex. In fact the intersection is the octahedron in T from part a). The union contains the vertices of the cube and the cube contains the union so the convex hull is the cube.

(d) (3 pts) Prove that there exists a sphere S passing through the midpoints of T and compute the spherical area of $S \cap T$.
The cube from the previous part must have side length side length $\frac{\sqrt{2}}{2}$ and by the symmetry $h_{O,-1}$ the midpoints of the edges of T are the midpoints of the faces of the cube. Therefore there exists a sphere passing through all these points, which is the inscribed sphere inside the cube. It has radius $r = \frac{\sqrt{2}}{4}$ so the total area of the sphere is $4\pi r^2 = \pi/2$. The part of S outside T is formed by four spherical caps, one on top of each of the four faces of T. Calculus shows that the area of a spherical cap of height h on a sphere of radius

r is $2\pi rh$. In this case the height h determined by $h = r - |Oc|$ where is the distance from O to the midpoint c of a face of T. Placing O at the Euclidean origin we see that the coordinates for c must be $c = \frac{r}{3}(1,1,1)$ as the face of T is part of the plane where $x+y+z=r$. It follows that $h = r - \frac{r}{\sqrt{3}}$ and so the final answer is $\pi/2 - 8\pi hr = \pi(\frac{1}{2} - 8r^2(1-\frac{1}{\sqrt{3}})) = \pi(\frac{1}{\sqrt{3}} - \frac{1}{2})$ which is about 15 percent of the area of S.

5. Hyperbolic plane. In this exercise we identify $z = x+iy \in \mathbb{C}$ with $(x,y) \in \mathbb{R}^2$ in the standard way. Likewise a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ is identified with a differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and multiplication by $f'(z)$ provides a linear map from $\mathbb{R}^2$ to itself that coincides with the derivative $\phi'(x,y)$. Next, the Euclidean inner product becomes $\langle v, w\rangle = \mathrm{Re}(v\bar{w})$.

   (a) (2 pts) For $a,b,c,d \in \mathbb{R}$ with $ad-bc=1$ set $f(z) = \frac{az+b}{cz+d}$. Check that $f'(z) = (cz+d)^{-2}$ and $\mathrm{Im}f(z) = \mathrm{Im}(z)|cz+d|^{-2}$.

$$f'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = (cz+d)^{-2}$$

   by assumption. Also

$$\mathrm{Im}f(z) = \frac{f-\bar{f}}{2i} = \frac{1}{2i}(\frac{(az+b)\overline{(cz+d)} - \overline{(az+b)}(cz+d)}{(cz+d)\overline{(cz+d)}} = \frac{1}{2i}\frac{(ad-bc)(z-\bar{z})}{|cz+d|^2} = \mathrm{Im}(z)|cz+d|^{-2}$$

   (b) (2 pts) Identifying the set $\{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ with the hyperbolic plane $\mathbb{H}$, show that the map f sends the hyperbolic plane to itself and that the hyperbolic metric g is written as $g(z)(v,w) = \frac{\mathrm{Re}(v\bar{w})}{(\mathrm{Im}z)^2}$.
   By the previous part $\mathrm{Im}f(z) = \mathrm{Im}(z)|cz+d|^{-2} > 0$ if $z \in \mathbb{H}$ so f sends $\mathbb{H}$ to itself. By definition and the identifications with the complex numbers the hyperbolic metric is written by the formula for g above.

   (c) (2 pts) Prove that the maps f (restricted to $\mathbb{H}$) are isometries in the sense that $g(f(z))(f'(z)v, f'(z)w) = g(z)(v,w)$.
   $f_{a,b,c,d}$ has inverse $f_{d,-b,-c,a}$ and both are differentiable because they are rational functions so f is a diffeomorphism. Next,

$$\mathrm{Re}(f'(z)v\overline{f'(z)w}) = \frac{1}{2}((cz+d)^{-2}v\overline{(cz+d)}^{-2}\bar{w} + \overline{(cz+d)}^{-2}\bar{v}(cz+d)^{-2}w)$$

$$= \frac{|cz+d|^{-4}}{2}(v\bar{w}+\bar{v}w) = |cz+d|^{-4}\mathrm{Re}(v\bar{w})$$

   Therefore

$$g(f(z))(f'(z)v, f'(z)w) = \frac{\mathrm{Re}(f'(z)v\overline{f'(z)w})}{(\mathrm{Im}f(z))^2} = \frac{|cz+d|^{-4}\mathrm{Re}(v\bar{w})}{(\mathrm{Im}z)^2|cz+d|^{-4}} = g(z)(v,w)$$

(d) (2 pts) Show that the straight line $Y = \{z \in \mathbb{H} : \text{Re}z = 0\}$ is a geodesic.
The reflection $\rho : \mathbb{H} \to \mathbb{H}$ defined by $\rho(a+bi) = -a+bi$ is an isometry because it is a Euclidean isometry and it preserves the y-coordinate of the points in the plane. As such it sends geodesics to geodesics. We know that for every point p and every vector v there exists a unique geodesic starting at p with velocity v at p. Consider for any $r \in \mathbb{R}$ the geodesic γ with $\gamma(0) = p = ri$ and $\gamma(0)' = v = i$. Since $\rho(\gamma)$ is also a geodesic with the same value at 0 and the same derivative at 0 we must have $\rho(\gamma) = \gamma$ by uniqueness. Therefore γ must be contained in Y. Again by uniqueness the whole of Y is part of a single geodesic.

Alternatively one may compute the Christoffel symbols to find (using real coordinates $z = (x, y)$):

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & -y^{-1} \\ -y^{-1} & 0 \end{pmatrix}$$

$$\Gamma^2_{ij} = \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}$$

so that the geodesic equations are

$$\gamma_1'' = 2\gamma_1'\gamma_2'/\gamma_2 \qquad \gamma_2'' = ((\gamma_2')^2 - (\gamma_1')^2)/\gamma_2$$

The line Y is parametrized with unit speed if we set $\gamma(t) = (0, e^t)$ and this choice of γ does indeed solve the above two equations.

(e) (2 pts) Show that if $c, d \neq 0$ then $f(Y)$ is a Euclidean semi-circle with radius $\frac{1}{2cd}$ and center $\frac{1}{2}(f(0) + f(\infty))$ where $f(\infty) = \frac{a}{c}$.
It since the proposed center is $\frac{1}{2}(f(0) + f(\infty)) = \frac{bc+ad}{2cd}$ it suffices to prove that for any real $t > 0$

$$|f(it) - \frac{bc + ad}{2cd}|^2 = \frac{1}{4c^2d^2}$$

Now $f(it) - \frac{bc+ad}{2cd} = \frac{2cd(ait+b)-(cit+d)(bc+ad)}{2cd(cit+d)} = \frac{(ad-bc)(cit-d)}{2cd(cit+d)} = \frac{cit-d}{2cd(cit+d)}$ implies

$$|f(it) - \frac{bc + ad}{2cd}|^2 = \frac{|cit - d|^2}{4c^2d^2|cit + d|^2} = \frac{1}{4c^2d^2}$$

as required.

(f) (2 pts) Explain why $f(Y)$ is a geodesic.
Isometries send geodesics to geodesics (Lemma 6.5) and it was shown in the previous parts that Y is a geodesic.

(g) (2 pts) Prove that all non-constant geodesics of $\mathbb{H}$ are Euclidean lines and semi-circles orthogonal to the real-axis.
For any point p and any vector v there is a unique Euclidean line or

semi-circle that is orthogonal to the real axis, passes through p and is tangent to the affine Euclidean line through p in the direction of vector v. So if I have a geodesic passing through p with velocity v then it must be part of that Euclidean semi-circle or line by uniqueness of geodesics.

(h) (2 pts) For every $\pi > \epsilon > 0$ construct a triangle whose angle sum is ϵ. Angle sum means sum of measures of geometric angles with respect to the Riemannian metric.
Since $g_{12} = 0$ for any point, the hyperbolic angles are exactly the Euclidean angles. Consider the Euclidean circles A, B, C with Euclidean radii $t, t, 1 + \frac{1}{t}$ and Euclidean centers $t-1, -t+1, 0 \in \mathbb{C}$. For $t > 2$ the geodesics bound a hyperbolic triangle and the sine of half the measure of the geometric angle opposite to C is $\frac{t-1}{t}$ which converges to 0 as t goes to infinity. The other two angles also go to 0 because A, B become tangent to C as t goes to infinity.

(i) (2 pts BONUS) Define $B = \{z \in \mathbb{H} : \mathrm{Re}(z) \in [0,1], |z - \frac{1}{2}| \geq \frac{1}{2}\}$. To emphasize the dependence of f on a, b, c, d we write f as $f_{a,b,c,d}$. Define $M = \{(a,b,c,d) \in \mathbb{Z} : ad - bc = 1\}$. Show that $\bigcup_{m \in M} f_m(B) = \mathbb{H}$ and that when $m \neq m'$ the intersection $f_m(B) \cap f_{m'}(B)$ is either a single geodesic or empty.
(This exercise is a little more difficult and will only be counted as bonus). It is clear that the images of B under the maps $f_{1,b,0,1}$ intersect in at most a geodesic when $b \in \mathbb{Z}$. Let us call these the big triangles. The big triangles also fill up all of $\mathbb{H}$ except for the Euclidean half disks with center $\frac{1}{2} + b$ and radius $\frac{1}{2}$. To show that the whole of $\mathbb{H}$ is covered by the f images of B it suffices to prove that any w inside the unit disk is covered. The map $j(z) = f_{0,1,-1,0}(z)$ maps the inside of the unit disk to the outside of the unit disk and vice versa. Notice that $\mathrm{Im} j(w) > \mathrm{Im}(w)$. This means that either $j(w)$ is already in the domain covered by the big triangles or we can translate it horizontally back to the unit circle and apply j once more. Eventually w will end up in the big region and so everything is covered by the images of B. The images intersect in at most a geodesic since any point of intersection can be mapped to a point inside the big triangle region and there we already know it is true.

6. (7 pts) In the proof of Lemma 6.5 check explicitly that Δ satisfies the axioms of an LC-connection as claimed.
We have an isometry f, which is a C^2 diffeomorphism $f : P \to Q$ that preserves the Riemannian inner products as in $g(f(p))(f'(p)v, f'(p)w) = g(v,w)(p)$. Throughout we will use the notation $p = f(q)$ and $q = f^{-1}(p)$. Recall that $f'X$ is a vector field on Q defined by $f'X(q) = f'(p)X(p)$. The formula makes it clear that if f is C^2 and X is C^2 differentiable then so is $f'X$. Also recall That $\partial_X Y$ is the vector field defined by $\partial_X Y(p) = Y'(p)X(p) = \sum_i (\partial_i Y)(p) X_i(p)$ whenever $X(p) = \sum_i X_i(p) e_i$.

Given an LC connection ∇ on Q we need to show that $\Delta_X Y = (f^{-1})'\nabla_{f'X} f'Y$ defines an LC connection on P as well. At least when X, Y are differentiable vector fields then $\nabla_{f'X} f'Y$ is a differentiable vector field on Q and so $\Delta_X Y$ defines a differentiable vector field on P.

We start by checking property 2) of LC connections, making use of the fact that ∇ satisfies the same axiom in Q. The key point is that $f'(uX) = (u \circ f^{-1})f'X$ for any vector field X and real valued function on P. The proof is just $f'(uX)(q) = f'(p)u(p)X(p) = u(p)f'(p)X(p) = (u \circ f^{-1})(q)(f'X)(q) = ((u \circ f^{-1})f'X)(q)$. Given vector fields X, Y, Z and function u on P we can now finish by

$$\Delta_{uX+Y} Z = (f^{-1})'\nabla_{f'uX+f'Y} f'Z = (f^{-1})'((u \circ f^{-1})\nabla_{f'X} f'Z + \nabla_{f'Y} f'Z) =$$

$$(f^{-1})'((u \circ f^{-1})\nabla_{f'X} f'Z) + (f^{-1})'\nabla_{f'Y} f'Z = (u \circ f^{-1} \circ f)(f^{-1})'\nabla_{f'X} f'Z + \Delta_Y Z = u\Delta_X Z + \Delta_Y Z$$

Next property 1):

$$\Delta_X(uY+Z) = (f^{-1})'\nabla_{f'X}(f'uY + f'Z) = (f^{-1})'\nabla_{f'X}((u \circ f^{-1})f'Y + f'Z) =$$

$$(f^{-1})'((u \circ f^{-1})\nabla_{f'X} f'Y + (\partial_{f'X}(u \circ f^{-1}))f'Y + \nabla_{f'X}(f'Z)) = u\Delta_X Y + \Delta_X Z + (f^{-1})'((\partial_{f'X}(u \circ f^{-1}))f'Y)$$

The last term can be simplified to

$$(f^{-1})'((\partial_{f'X}(u \circ f^{-1}))f'Y) = (\partial_{f'X}(u \circ f^{-1}) \circ f)Y$$

and even further by the chain rule using

$$\partial_{f'X}(u \circ f^{-1})(q) = (u \circ f^{-1})'f'X(q) = (u \circ f^{-1})'f'(p)X(p) = (u \circ f^{-1} \circ f)'(p)X(p) = u'(p)X(p) = (\partial_X u)(p)$$

as required.

Property 4 is easiest to see by writing out $X = \sum_i X_i e_i$ and $Y = \sum_j Y_j e_j$. By the same property for ∇ we find

$$\Delta_X Y - \Delta_Y X = (f^{-1})'(\nabla_{f'X} f'Y - \nabla_{f'Y} f'X) = (f^{-1})'(\partial_{f'X} f'Y - \partial_{f'Y} f'X)$$

so we proceed to investigate the first of the final two terms. If we set $H(s) = f'Y \circ f(s) = f'(s)Y(s) = \sum_j (\partial_j f)(s)Y_j(s)$ then the chain rule says

$$\partial_{f'X} f'Y(q) = (f'Y)'(q)(f'X)(q) = (f'Y)'(q)f'(p)X(p) = H'(p)X(p) =$$

$$\sum_i (\partial_i H)(p)X_i(p) = \sum_{i,j} (\partial_i(\partial_j f)Y_j)(p)X_i(p) = \sum_{i,j} (\partial_i \partial_j f)(p)Y_j(p)X_i(p) + (\partial_j f)(p)(\partial_i Y_j)(p)X_i(p)$$

Subtracting from this $\partial_{f'X} f'Y(q)$ gives

$$\sum_{i,j} (\partial_j f)(p)(\partial_i Y_j)(p)X_i(p) - (\partial_j f)(p)(\partial_i X_j)(p)Y_i(p) = f'\partial_X Y(q) - f'\partial_Y X(q)$$

because $f'\partial_X Y(q) = f'(p)\partial_X Y = f'(p)Y'(p)X(p)$. Putting it all together this proves the property after applying $(f^{-1})'$.

Finally property 3) we prove in the abbreviated form

$$\langle \Delta_X Y, Z\rangle + \langle Y, \Delta_X Z\rangle = \partial_X \langle Y, Z\rangle$$

Using the fact that f is an isometry so that $\langle f'Y, f'Z\rangle = \langle Y, Z\rangle$ we simplify the left hand side to

$$\langle \Delta_X Y, Z\rangle + \langle Y, \Delta_X Z\rangle = \langle (f^{-1})'\nabla_{f'X} f'Y, Z\rangle + \langle Y, (f^{-1})'\nabla_{f'X} f'Z\rangle =$$

$$\langle \nabla_{f'X} f'Y, f'Z\rangle + \langle f'Y, \nabla_{f'X} f'Z\rangle = \partial_{f'X}\langle f'Y, f'Z\rangle$$

Finally the isometry property says $\langle f'Y, f'Z\rangle(q) = \langle Y, Z\rangle(p)$ so

$$\partial_{f'X}\langle f'Y, f'Z\rangle(q) = \langle f'Y, f'Z\rangle'(q) f'X(q) = \langle f'Y, f'Z\rangle'(q) f'(p)X(p)$$

$$= \langle Y, Z\rangle'(p) f'(p)X(p) = \partial_X \langle Y, Z\rangle(p)$$

finishing the proof.

7. In this exercise we study Riemannian charts of the form $(P, \phi^* g_E)$ where $\phi : P \to \mathbb{R}^3$ is injective and C^2 with $P \subset \mathbb{R}^2$ some open set such that $\forall p \in P$: the derivative $\phi'(p)$ is injective.

   (a) (4 pts) If $r : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine Euclidean isometry and $\psi = r \circ \phi$ then prove that there is an isometry from $(P, \phi^* g_E)$ to $(P, \psi^* g_E)$. Since all Euclidean isometries of $\mathbb{R}^3$ are the composition of a translation and a linear isometry, notice that the derivative of r' of r is also an Euclidean isometry which is a linear map. The identity $id : P \to P$ is such an isometry since by the chain rule

   $$\phi^* g_E(p)(v, w) = g_E(\phi'(p)v, \phi'(p)w) = g_E(r' \circ \phi'(p)v, r' \circ \phi'(p)w)$$

   $$= g_E((r\circ\phi)'(p)v, (r\circ\phi)'(p)w) = g_E(\psi'(p)v, \psi'(p)w) = \psi^* g_E(p)(v, w)$$

   (b) (3 pts) Suppose $\phi(x, y) = (x, y, f(x, y))$. Compute the scalar curvature of the Riemannian chart $(P, \phi^* g_E)$ at $(0, 0)$ in the case $f(x, y) = x^2 - y^2$. Notice that this actually follows from the next part because $f(x, y) = x^2 - y^2$ satisfies the conditions $\partial_1 f(0, 0) = \partial_2 f(0, 0) = 0$. In this case the Hessian at $(0, 0)$ is $\partial_1\partial_1 f(0, 0)\partial_2\partial_2 f(0, 0) - \partial_1\partial_2 f(0, 0)\partial_2\partial_1 f(0, 0) = -4$ so the scalar curvature is -8.

   (c) (4 pts) Prove for general C^2-functions $f : P \to \mathbb{R}$ such that $\partial_1 f(0, 0) = \partial_2 f(0, 0) = 0$ that the scalar curvature of the Riemannian chart $(P, \phi^* g_E)$ at point $(0, 0)$ is $2\text{Hess}(f)(0, 0)$. Here the Hessian is $\text{Hess}(f) = \det(\partial_i\partial_j f)$ is the determinant of the matrix of second partial derivatives.

We start by computing the Jacobian matrix for

$$\phi'(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 f & \partial_2 f \end{pmatrix}$$

Taking inner products of the columns we find that the matrix for the Riemannian metric g is

$$(g_{ij}) = \begin{pmatrix} 1 + (\partial_1 f)^2 & (\partial_1 f)(\partial_2 f) \\ (\partial_1 f)(\partial_2 f) & 1 + (\partial_2 f)^2 \end{pmatrix}$$

and the inverse matrix is using $d = 1 + (\partial_1 f)^2 + (\partial_2 f)^2$

$$(g_{ij}^{-1}) = \frac{1}{d} \begin{pmatrix} 1 + (\partial_2 f)^2 & -(\partial_1 f)(\partial_2 f) \\ -(\partial_1 f)(\partial_2 f) & 1 + (\partial_1 f)^2 \end{pmatrix}$$

Next one finds that the Christoffel symbols Γ_{ij}^k are conveniently expressed in terms of the two matrices (Γ_{ij}^1) and (Γ_{ij}^2) as:

$$(\Gamma_{ij}^1) = \frac{(\partial_1 f)((1 + (\partial_2 f)^2 - (\partial_1 f)(\partial_2 f))}{d} \mathrm{Hess}(f)$$

$$(\Gamma_{ij}^2) = \frac{(\partial_2 f)(1 + (\partial_1 f)^2 - (\partial_1 f)(\partial_2 f))}{d} \mathrm{Hess}(f)$$

Recall that the coefficients of the Riemann curvature tensor are

$$R_{i,j,k}^\ell = \partial_i \Gamma_{jk}^\ell - \partial_j \Gamma_{ik}^\ell + \sum_r \Gamma_{jk}^r \Gamma_{ir}^\ell - \Gamma_{ik}^r \Gamma_{jr}^\ell$$

evalutated at $(0,0)$ the final sum of terms drops out since $\Gamma_{ij}^k(0,0) = 0$ for any i, j, k. Therefore the Ricci curvature at $(0,0)$ is given by

$$R_{jk}(0,0) = \sum_\ell R_{\ell,j,k}^\ell(0,0) = \sum_\ell \partial_\ell \Gamma_{jk}^\ell(0,0) - \partial_j \Gamma_{\ell k}^\ell(0,0)$$

Also notice that in evaluating these derivatives at $(0,0)$ the only option is to take the derivative of the $(\partial_k f)$ part in front of the formula for Γ_{ij}^k If we derive some other part of the formula this first derivative will yield 0. More specifically we find

$$\partial_s \Gamma_{ij}^k(0,0) = \mathrm{Hess}_{sk}(f)(0,0)\mathrm{Hess}_{ij}(f)(0,0)$$

It follows that $R_{jk} = \det \mathrm{Hess}(f)(0,0)$ if $j = k$ and 0 otherwise. The scalar curvature is computed from the Ricci curvature by the formula $S = \sum_{i,j} g_{ij}^{-1} R_{ij}$ and since g_{ij}^{-1} is the identity matrix we obtain $S(0,0) = 2 \det \mathrm{Hess}(f)(0,0)$ finishing the proof.

(d) (4 pts) Define the (thick) Gauss map $G : P \times (-1, 1) \to \mathbb{R}^3$ sending a point in the thickened chart to the normal vector to the image of ϕ by

$$G(p, t) = (t + 1)\frac{\partial_1\phi(p) \times \partial_2\phi(p)}{|\partial_1\phi(p) \times \partial_2\phi(p)|}$$

Traditionally curvature is approached by studying how fast the normal vector turns. This is captured by the Gauss curvature at p is $\det G'(p, 0)$. Under the same assumptions as in part c) prove that to the scalar curvature at $(0, 0)$ equals twice the Gauss curvature at $(0, 0)$.

By the previous part it suffices to show that the Gauss curvature at $p = (0, 0)$ which is $G'(0, 0, 0)$ satisfies

$$\det G'(0, 0, 0) = \det \mathrm{Hess}(f)(0, 0)$$

First we compute that with $d = 1 + (\partial_1 f)^2 + (\partial_2 f)^2$ and $p = (u, v)$ we have

$$G(u, v, t) = \frac{t + 1}{d}(-\partial_1 f, -\partial_2 f, 1)$$

To evaluate the matrix of partial derivatives at $(0, 0, 0)$ we notice we should always be differentiating the $\partial_i f$ part because otherwise it will vanish, also $\frac{t+1}{d}(0, 0, 0) = 1$. Therefore the Jacobian matrix for $G'(0, 0, 0)$ is

$$\begin{pmatrix} -\partial_1\partial_1 f & -\partial_2\partial_1 f & 0 \\ -\partial_1\partial_2 f & -\partial_2\partial_2 f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expanding the final column it is clear that

$$\det G'(0, 0, 0) = \det \mathrm{Hess}(f)(0, 0)$$

as required.